

# ON THE ORBITAL STABILITY OF STANDING-WAVES SOLUTIONS TO A COUPLED NON-LINEAR KLEIN-GORDON EQUATION

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**ABSTRACT.** We consider a system of two coupled non-linear Klein-Gordon equations. We show the existence of standing waves solutions and the existence of a Lyapunov function for the ground state.

## INTRODUCTION

The purpose of this work is to set some basic results to prove the orbital stability of standing-waves solutions to a coupled non-linear Klein-Gordon equation

$$(NLKG) \quad \begin{cases} D_{tt}\phi_1 - \Delta\phi_1 + m_1^2\phi_1 + D_1F(\phi) = 0 \\ D_{tt}\phi_2 - \Delta\phi_2 + m_2^2\phi_2 + D_2F(\phi) = 0. \end{cases}$$

The existence of standing-waves is obtained through a variational approach which provides with a solution  $(u, \omega)$  of the elliptic system

$$(ES) \quad \begin{cases} -\Delta u_1 + m_1^2 u_1 + D_1F(u) = \omega_1^2 u_1 \\ -\Delta u_2 + m_2^2 u_2 + D_2F(u) = \omega_2^2 u_2. \end{cases}$$

For scalar field equations, the development of tools for rigorous proofs of the orbital stability of standing-waves for the nonlinear Klein-Gordon equation, or the Schrödinger equation is relatively recent. The first results for NLKG are due to J. Shatah in [21] (and generalised to coupled NLKG in [25]) where it is shown that solutions of

$$(0.3\omega) \quad -\Delta u + (1 - \omega^2)u + F'(u) = 0$$

which are minimizers of some functional  $J_\omega$  on a natural constraint  $M_\omega$ , are stable for the values of  $\omega$  where the function  $\omega \mapsto \inf_{M_\omega} J_\omega$  is convex; for the NLS (and other scalar field equations), in the work of T. Cazenave and P. L. Lions [11] it is proved the orbital stability of

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solutions obtained as minimizers of the energy functional  $\mathcal{E}$ , [11, p. 3], on the constraint

$$N_\lambda := \{\|u\|_{L^2(\mathbb{R}^n)}^2 = \lambda\}.$$

This result has been generalised to a large class of non-linearities for NLS in [4] and in [20] for some class of coupled NLS. In [3] the orbital stability of NLKG for a class of solutions (in general different from [21]) obtained as minimizers of the functional

$$E(u, \omega) = \frac{1}{2} \int_{\mathbb{R}^n} (|Du|^2 + m^2 u^2) + \int_{\mathbb{R}^n} F(u) + \frac{\omega^2}{2} \int_{\mathbb{R}^n} u^2$$

on the constraint

$$M_C := \{(u, \omega) \mid \omega \|u\|_{L^2(\mathbb{R}^n)}^2 = C\}$$

is considered. The hypotheses on the non-linearity  $F$  are very general and the stability is proved under the assumption that local solutions of NLKG exist in  $H^1$  and radially symmetric minimizers are isolated in  $M_C$ .

This work deals with standing-waves solutions obtained as minimizer of the energy functional

$$(E) \quad E(u, \omega) = \frac{1}{2} \sum_{j=1}^2 \int_{\mathbb{R}^n} (|Du_j|^2 + m_j^2 u_j^2 + \omega_j^2 u_j^2) + \int_{\mathbb{R}^n} F(u)$$

on the constraint

$$(C) \quad M_C := \{(u, \omega) \mid \omega_j \|u_j\|_{L^2(\mathbb{R}^n)}^2 = C_j\}.$$

The utility of this variational setting is two-fold: firstly, the Euler-Lagrange equations correspond to a solution of (ES), thus we do not need a further discussion on the sign of the Lagrange multipliers. Moreover, due to the symmetry of the Lagrangian of (NLKG), for a smooth solution  $\phi$  we have the conservation laws

$$(E) \quad \mathbf{E}(\phi, \phi_t) = \frac{1}{2} \int_{\mathbb{R}^n} |\phi_t|^2 + |D\phi|^2 + 2V(\phi)$$

$$(C) \quad \mathbf{C}_j(\phi, \phi_t) = -\text{Im} \int_{\mathbb{R}^n} \phi_t^j \bar{\phi}_j(t, \cdot), \quad 1 \leq j \leq 2.$$

These correspond to  $E$  and  $C_j$  on standing-waves solutions. The main theorems are the following:

**Theorem A.** *Given a minimizing sequence  $(u_n, \omega_n)$  for  $E$  over  $M_C$ , there exists a minimizer  $(u, \omega)$  and  $(y_n)_{n \geq 1} \subset \mathbb{R}^n$  such that, up to extract a subsequence*

$$u_n^j = u_j(\cdot + y_n) + o(1) \text{ in } H^1(\mathbb{R}^n), \quad \omega_n \rightarrow \omega \text{ in } \mathbb{R}^2$$

for  $1 \leq j \leq 2$ .

As in the scalar case in [3], compactness of minimizing sequences is proved for the simpler functional

$$J(u) = \frac{1}{2} \sum_{j=1}^2 \int_{\mathbb{R}^n} |Du_j|^2 + \int_{\mathbb{R}^n} F(u).$$

on the constraint

$$N_\rho := \{u \mid \|u_j\|_{L^2(\mathbb{R}^n)}^2 = \rho\}.$$

In the scalar case, the compactness of the minimizing sequences of  $J$ , [4], is achieved by proving the sub-additivity property of  $I_\rho := \inf_{N_\rho} J$ , that is

$$I_\rho < I_\tau + I_{\rho-\tau}, \quad 0 < \tau < \rho.$$

We follow the same approach. However, while in the scalar case, such inequality can be proved by rescaling two minimizing sequences in  $N_\tau$  and  $N_{\rho-\tau}$ , a more effort is needed for systems; we address this property to Section 4. In Lemma 4 we show that there exists  $D > 0$ , depending only on  $\rho$  and  $\tau$  such that

$$I_\rho < I_\tau + I_{\rho-\tau} - D.$$

The inequality uses the symmetric decreasing rearrangement, [16]. The idea we follow is that, if two bumps  $u \in N_\tau$  and  $v \in N_{\rho-\tau}$  have small interaction, then

$$\|Dw^*\|^2 < \|Du\|^2 + \|Dv\|^2 - D$$

where  $w = u + v$  and  $w^*$  is the symmetric rearrangement. The second theorem concerns the properties of two subsets of the phase space of (NLKG),  $H^1(\mathbb{R}^n, \mathbb{C}^2) \oplus L^2(\mathbb{R}^n, \mathbb{C}^2)$ . To a minimizer  $(u, \omega)$  of  $E$  over  $M_C$  we can associate

$$\Gamma(u, \omega) = \begin{cases} (\lambda u(\cdot + y), -i\omega \lambda u(\cdot + y)) \\ (\lambda_1, \lambda_2, y) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^n, \quad |\lambda_1| = |\lambda_2| = 1 \end{cases},$$

and to a constraint  $M_C$ , we can associate

$$(GS) \quad \Gamma_C = \bigcup \{ \Gamma(u, \omega) \mid (u, \omega) \in K_C \},$$

called *ground state*, where

$$m_C := \inf_{M_C} E, \quad K_C := \{(u, \omega) \mid E(u, \omega) = m_C\}.$$

**Theorem B.** *Given a sequence*

$$(\Phi_n)_{n \geq 1} \subset H^1(\mathbb{R}^n, \mathbb{C}^2) \oplus L^2(\mathbb{R}^n, \mathbb{C}^2),$$

*then  $d(\Phi_n, \Gamma_C) \rightarrow 0$  if and only if*

$$\mathbf{E}(\Phi_n) \rightarrow m_C, \quad \mathbf{C}_j(\Phi_n) \rightarrow C_j.$$

*for  $1 \leq j \leq 2$ .*

A proof of this theorem in the scalar case can be found in [3] under the assumption that the NLKG is locally well-posed. In our proof we drop this assumption. The keypoint of the proof lies in the following property: given  $\phi \in H^1(\mathbb{R}^n, \mathbb{C})$  such that  $|\phi| > 0$  everywhere and

$$\|D\phi\| = \|D|\phi|\|$$

there exists  $\lambda \in \mathbb{C}$  such that  $|\lambda| = 1$  and  $\phi = \lambda|\phi|$ . We show this in Lemma 4 for  $H^1(\mathbb{R}^n, \mathbb{R}^m)$  and  $m \geq 1$ . A similar property is shown in [17, Theorem 7.8] under the stronger assumption that  $|\phi_j| > 0$  for some  $1 \leq j \leq m$ .

The non-linear term  $F$  is assumed to be continuously differentiable with subcritical growth and can be written as non-negative perturbation of a coupling term

$$F(u) = -\beta|u_1 u_2|^\gamma + G, \quad G \geq 0.$$

Theorems A and B are addressed to the proof that  $\Gamma_C$  is a stable subset of the phase space

$$X := H^1(\mathbb{R}^n, \mathbb{C}^2) \oplus L^2(\mathbb{R}^n, \mathbb{C}^2).$$

Thus, it is very natural to ask whether we have local existence of solutions to (NLKG) with initial data in  $X$ . However, from known results on the non-linear scalar wave equation, we can expect local existence only

$$X_k := H^k(\mathbb{R}^n, \mathbb{C}^2) \oplus H^{k-1}(\mathbb{R}^n, \mathbb{C}^2), \quad k > n/2$$

with the general assumptions we make on  $F$ . Moreover, for  $k = 1$  even conservation laws **(E)** and **(C)** are not known to hold for every non-linearity. In order to obtain the stability of  $\Gamma(u, \omega)$  it seems that the non-degeneracy condition

$$\{(u(\cdot + y), \omega) \mid y \in \mathbb{R}^n\} \text{ is isolated in } K_C$$

is rather necessary. We do not tackle in this work the problem of the existence of local solutions and the non-degeneracy condition.

Numerical results on the existence of standing-waves have been obtained in [9] when  $n = 3$  and critical exponents.

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## 1. REGULARITY PROPERTIES

We fix  $n \geq 3$  and recall the well-known inequalities for a function  $u \in H^1(\mathbb{R}^n)$

$$\begin{aligned} (1) \quad & \|u\|_{L^{2^*}} \leq S \|Du\|_{L^2} \\ (2) \quad & \|u\|_{L^p} \leq \|u\|_{L^2} + S \|Du\|_{L^2} \end{aligned}$$

for some  $S > 0$  and for every  $2 \leq p \leq 2^*$  and  $2^* = 2n/n - 2$ , check [8, Corollaire IX.10,p. 165]. Given an integer  $m$ , we set

$$H := \bigoplus_{k=1}^m H^1(\mathbb{R}^n).$$

Given  $u \in L^p(\mathbb{R}^n, \mathbb{R}^m)$ , we also set

$$\|u\|_p = \|u\|_{L^p}, \quad \|u\| = \|u\|_{L^2}.$$

On  $H$  we consider the norm defined as

$$\|u\|_H^2 := \sum_{j=1}^m \|u_j\|^2 + \|Du_j\|^2.$$

**Definition 1.** A real-valued function  $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a *combined power-type* if there exists a constant  $c > 0$  and  $p \leq q$  such that

$$|F(u)| \leq c(|u|^p + |u|^q)$$

for every  $u \in \mathbb{R}^n$ . If  $p = q$ , we say that  $F$  is a *power-type*.

**Proposition 1.** Let  $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a differentiable function such that  $F(0) = 0$  and there are  $p \leq q$  such that

$$(3) \quad |DF(u)| \leq c_1(|u|^{p-1} + |u|^{q-1}).$$

Then, there are  $F_p, F_q: \mathbb{R}^n \rightarrow \mathbb{R}^m$  differentiable such that  $F_p(0) = F_q(0) = 0$  and

$$F = F_p + F_q$$

$$|DF_p(u)| \leq c_{p-1}|u|^{p-1}, \quad |DF_q(u)| \leq c_{q-1}|u|^{q-1}$$

*Proof.* Because  $F(0) = 0$ , and by the Fundamental Theorem of the Calculus, it follows from the hypotheses that

$$(4) \quad |F(u)| \leq c_0(|u|^p + |u|^q)$$

in fact  $c_0$  could be chosen to be  $\sqrt{m}c_1/p$ . Let  $\eta \in C^1(\mathbb{R}^m, \mathbb{R})$  be a non-negative function such that

$$\eta(u) = \begin{cases} 1 & \text{if } |u| \leq 1 \\ 0 & \text{if } |u| \geq 2 \end{cases}$$

with  $\eta \leq 1$  and  $|D\eta| \leq 2$ . On  $B(0, 2)$ , by (3), we have

$$\begin{aligned} |\eta DF(u)| &\leq c_1(|u|^{p-1} + |u|^{q-1}) = c_1[|u|^{p-1} + 2^{q-1}(|u|/2)^{q-1}] \\ &\leq c_1(|u|^{p-1} + 2^{q-p}|u|^{p-1}) = c_1(1 + 2^{q-p})|u|^{p-1}. \end{aligned}$$

The second inequality follows from  $p \leq q$ . Because  $\eta$  vanishes outside  $B(0, 2)$  the inequality above holds in  $\mathbb{R}^m$ . On the annulus  $C(1, 2)$  we have

$$|FD\eta| \leq 2c_0(|u|^{p-1} + |u|^{q-1}) \leq 2c_0(1 + 2^{q-p-2})|u|^{p-1}.$$

Since  $D\eta$  vanishes outside  $C(1, 2)$ , the inequality above holds in  $\mathbb{R}^m$ . Combining the last two inequalities, we prove that  $D(\eta F)$  is power-type. Similarly, one shows that

$$|D(1 - \eta)F| \leq 2(c_1 + 2c_0)|u|^{q-1}.$$

We set  $F_p := \eta F$  and  $F_q := (1 - \eta)F$ . Thus,

$$(5) \quad F = F_p + F_q$$

is the desired decomposition.  $\square$

Let  $F$  be a real-valued continuously differentiable function on  $\mathbb{R}^m$  such that

$$(6) \quad |DF(u)| \leq c_1(|u|^{p-1} + |u|^{q-1}), \quad F(0) = 0, \quad 2 \leq p \leq q \leq 2^*$$

for every  $u \in \mathbb{R}^m$ . For every  $u \in H$  we have  $F(u) \in L^1(\mathbb{R}^n)$  from inequalities (2) and (4). Therefore we have a well-defined functional

$$J: H \rightarrow \mathbb{R}$$

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^n} |Du|^2 + \int_{\mathbb{R}^n} F(u).$$

**Proposition 2.** *The functional  $J$  defined above satisfies the following:*

- (a)  *$J$  is of class  $C^1(H, \mathbb{R})$ ,*
- (b) *if  $q < 2^*$ , given a weakly converging sequence  $u_n \rightharpoonup u$  in  $H$ , up to extract a subsequence, we have*

$$J(u_n - u) = J(u_n) - J(u) + o(1).$$

The proof of (a) uses the same technique of [1, Theorem 2.2] and [1, Theorem 2.6] which deals with bounded domains. In fact, such restriction is not necessary; the proof of (b) is the same as the scalar case of [4, Appendix].

*Proof.* (a). Since the map  $u \mapsto \|Du\|_2^2$  is smooth on  $H$ , we only need to prove that

$$\mathcal{F}(u) := \int_{\mathbb{R}^n} F(u)$$

is  $C^1(H)$ . Moreover, by Proposition 1, we can suppose that  $F$  and  $DF$  are power type non-linearities and

$$(7) \quad |DF(u)| \leq c_1|u|^{p-1}, \quad |F(u)| \leq c_0|u|^p.$$

From the first of the two inequalities above,  $|DF(u)|$  is in  $L^{p'}$ , where  $p' = p/(p-1)$ . Because  $|D_j F(u)| \leq |DF(u)|$  the application

$$(8) \quad \mathcal{G}_j: H^1 \rightarrow L^{p'}, \quad u \mapsto D_j F(u).$$

is well defined for every  $1 \leq j \leq m$ . We prove that it is also continuous. To this end, let  $u_n \rightarrow u$  be a converging sequence in  $H$ ; we show

that  $\mathcal{G}_j(u_n)$  has a converging subsequence and that all the converging subsequences have the same limit  $\mathcal{G}_j(u)$ . Thus,

$$\mathcal{G}_j(u_n) \rightarrow \mathcal{G}_j(u).$$

Up to extract a subsequence, we can suppose that there exists  $v \in L^p(\mathbb{R}^n)$  such that

$$u_n \rightarrow u, \quad |u_n^j| \leq v$$

almost everywhere, for  $1 \leq j \leq m$ . Because  $D_j F$  is continuous, by the convergence above, we have

$$D_j F(u_n) \rightarrow D_j F(u) \text{ pointwise a.e.}$$

$$|D_j F(u_n) - D_j F(u)|^{p'} \leq (2c_1)^{p'} |v|^p \in L^1(\mathbb{R}^n).$$

Thus, by the dominate convergence theorem, we obtain the convergence of  $\mathcal{G}_j(u_n)$  to  $\mathcal{G}_j(u)$ . Now, for every  $u \in H$ , we consider the linear functional

$$(9) \quad L_u: H \rightarrow \mathbb{R}$$

$$(10) \quad L_u(\varphi) := \sum_{j=1}^m \int_{\mathbb{R}^n} D_j F(u) \varphi_j,$$

which is well-defined and bounded by the Hölder inequality. Next, we show that

$$\mathcal{F}(u + \varphi) - \mathcal{F}(u) - L_u(\varphi) = o(\varphi).$$

We prove the convergence above on sequences  $\varphi_n \rightarrow 0$ . The left term equals

$$\begin{aligned} & \int_{\mathbb{R}^n} F(u + \varphi_n) - F(u) - L_u(\varphi_n) \\ &= \int_0^1 \int_{\mathbb{R}^n} \langle DF(u + t\varphi_n), \varphi \rangle - L_u(\varphi_n) \\ &= \sum_{j=1}^m \int_0^1 \int_{\mathbb{R}^n} (D_j F(u + t\varphi_n) - D_j F(u)) \varphi_n^j. \end{aligned}$$

Thus, by the definition of  $\mathcal{G}_j$  and the Hölder inequality, we obtain

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} F(u + \varphi_n) - F(u) - L_u(\varphi_n) \right| \\ & \leq \sum_{j=1}^m \|\varphi_n^j\|_p \int_0^1 \|\mathcal{G}_j(u + t\varphi_n) - \mathcal{G}_j(u)\|_{p'} \\ & \leq \sqrt{1 \vee S} \|\varphi_n\|_H \left[ \sum_{j=1}^m \left( \int_0^1 \|\mathcal{G}_j(u + t\varphi_n) - \mathcal{G}_j(u)\|_{p'} \right)^2 \right]^{1/2} \end{aligned}$$

where the second inequality still follows from (2) and the Schwarz inequality for the euclidean product on  $\mathbb{R}^m$ . By the continuity of  $\mathcal{G}_j$ , the functions

$$g_n^j(t) := \|\mathcal{G}_j(u + t\varphi_n) - \mathcal{G}_j(u)\|_{p'}$$

are continuous on the unit interval and converge pointwise. Because the sequence  $\varphi_n$  is bounded in  $H$ , they are also uniformly bounded from above. Then by the dominated convergence theorem, we have

$$\int_0^1 g_n^j(t) dt \rightarrow 0.$$

Thus, the last term of the previous inequality is  $o(1)\|\varphi\|_H$  which proves that  $\mathcal{F}$  is continuous and differentiable in  $u$  and

$$D\mathcal{F}(u) = L_u.$$

Finally, we observe that by the continuity of  $\mathcal{G}_j$  and the definition of  $L_u$  in (10), the map

$$D\mathcal{F}: H \rightarrow \mathcal{L}(H, H^*)$$

is continuous. Thus,  $\mathcal{F} \in C^1(H, \mathbb{R})$ . Then,  $J \in C^1(H, \mathbb{R})$ .

(b). For  $u \mapsto \|Du\|^2$  the property follows easily from

$$\begin{aligned} \|D(u_n - u)\|^2 &= \|Du_n\|^2 + \|Du\|^2 - 2(Du_n, Du) \\ &= \|Du_n\|^2 - \|Du\|^2 + o(1). \end{aligned}$$

Again, from Proposition 1 we can suppose that (7) holds. We set  $v_n := u_n - u$ . Let us fix  $\varepsilon > 0$ . We prove that there exists a subsequence of  $(v_n)$  such that

$$\lim_{n \rightarrow +\infty} |\mathcal{F}(u + v_n) - \mathcal{F}(u) - \mathcal{F}(v_n)| < \varepsilon.$$

Given  $R > 0$ , we have

$$\begin{aligned} &\mathcal{F}(u + v_n) - \mathcal{F}(u) - \mathcal{F}(v_n) \\ &= \int_{\mathbb{R}^n} F(u + v_n) - F(u) - \int_{\mathbb{R}^n} F(v_n) \\ &= \int_{B_R} F(u + v_n) - F(u) - \int_{B_R} F(v_n) \\ &\quad + \int_{B_R^c} F(u + v_n) - F(v_n) - \int_{B_R^c} F(u) =: A + B \end{aligned}$$

We estimate separately the summands of the last term of the equality. Since  $F$  differentiable, we have

$$\begin{aligned} B &= \int_{B_R^c} F(u + v_n) - F(v_n) - \int_{B_R^c} F(u) \\ &= \int_{B_R^c} \int_0^1 \langle DF(v_n + tu), u \rangle dt - \int_{B_R^c} F(u) =: B_1 + B_2. \end{aligned}$$



By (7) and the Hölder inequality, we have

$$\begin{aligned} |B_1| &\leq c_1 \int_{B_R^c} |u + tv_n|^{p-1} |u| \leq c_1 \sup_{n,t} \|u + tv_n\|_{L^p(B_R^c)}^{p-1} \|u\|_{L^p(B_R^c)} \\ |B_2| &\leq c_0 \|u\|_{L^p(B_R^c)}^p. \end{aligned}$$

Because  $v_n$  converges weakly, the supremum above is finite. Since  $u \in L^p(\mathbb{R}^n)$ , there exists  $R(\varepsilon)$  such that  $|B_i| \leq \varepsilon/4$ . Similarly, we have

$$\begin{aligned} A &= \int_{B_{R(\varepsilon)}} F(u + v_n) - F(u) - \int_{B_{R(\varepsilon)}} F(v_n) \\ &= \int_{B_{R(\varepsilon)}} \int_0^1 \langle DF(u + tv_n), v_n \rangle - \int_{B_{R(\varepsilon)}} F(v_n) =: A_1 + A_2. \end{aligned}$$

From (7), we have

$$\begin{aligned} |A_1| &\leq c_1 \int_{B_{R(\varepsilon)}} |u + tv_n|^{p-1} |v_n| \leq c_1 \sup_{n,t} \|u + tv_n\|_{L^p(B_{R(\varepsilon)})}^{p-1} \|v_n\|_{L^p(B_{R(\varepsilon)})} \\ |A_2| &\leq c_0 \|v_n\|_{L^p(B_{R(\varepsilon)})}^p. \end{aligned}$$

Because  $p < 2^*$  the inclusion of  $L^p(B_{R(\varepsilon)})$  in  $H^1(B_{R(\varepsilon)})$  is compact, [8, Théorème IX.16, p. 169]. Thus, we can extract a subsequence such that  $v_n \rightarrow 0$  in  $L^p(B_{R(\varepsilon)})$ . If we choose  $n$  large enough, we obtain  $|A_i| \leq \varepsilon/4$ .

If we repeat the same argument for  $\varepsilon_k = 1/k$  we obtain subsequences

$$(v_{n,k}) \subseteq \cdots \subseteq (v_{n,2}) \subseteq (v_{n,1}).$$

Let  $n_k$  be such that

$$|\mathcal{F}(u + v_{n_k,k}) - \mathcal{F}(u) - \mathcal{F}(v_{n_k,k})| < \frac{1}{k}.$$

Thus  $w_k := v_{n_k,k}$  is a subsequence of  $(v_n)$ . Then, the sequence  $u_{n_k} := w_k + u$  is a subsequence of  $(u_n)$  and satisfies the required properties.  $\square$

## 2. THE VARIATIONAL SETTING

Throughout this and the next sections we assume  $m = 2$  and the following properties on  $F$ :

- (A1)  $F(u) = -\beta |u_1 u_2|^\gamma + G(u), \quad 1 < \gamma < 1 + 2/n,$
- (A2)  $|DG(u)| \leq c_1(|u|^{p-1} + |u|^{q-1}), \quad 2\gamma < p \leq q < 2^*,$
- (A3)  $G(u) = G(|u_1|, |u_2|), \quad G \geq 0.$

Finally, we assume that  $G$  is well-behaved with respect to the Steiner rearrangement. That is, given  $u_1, u_2 \in H^1(\mathbb{R}^n)$ , and denoting by  $u_1^*$  and  $u_2^*$  their Steiner symmetric rearrangements (check [16]), we have

$$(A4) \quad \int_{\mathbb{R}^n} G(u_1^*, u_2^*) \leq \int_{\mathbb{R}^n} G(u_1, u_2).$$

The assumptions (A1,A2) are the natural extension of the hypothesis  $(F_p, F_0)$  and  $(F_2)$  made in the scalar case by V. Benci, M. Ghimenti *et al.*, [4]. The proof of the next Lemma, which we include for the sake of completeness, is similar to the ones of [4, Lemma 5, Proposition 7].

**Lemma 1.** *For every  $\rho \in \mathbb{R}^2$  with  $\rho_j > 0$ , we have*

- (i)  $\inf_{N_\rho} J =: I_\rho$  is finite and negative,
- (ii) minimizing sequences of  $J$  on  $N_\rho$  are bounded,

*Proof.* (i). Let  $v \in H^1$  be such that  $\|Dv\| = \|v\| = 1$ . Given  $s_i, R > 0$ , we define

$$u(x) = (s_1 v(x/R), s_2 v(x/R)).$$

By a change of variable, it can be easily checked that

$$\|u_j\|^2 = s_j^2 R^n, \quad \|Du_j\|^2 = s_j^2 R^{n-2}.$$

We choose  $R$  and  $s$  such that  $\rho_j = s_j^2 R^n$ . Thus,  $s_2 = \lambda s_1$ , where  $\lambda\sqrt{\rho_1} := \sqrt{\rho_2}$ . We have

$$\|Du\|^2 = \frac{1}{2} (s_1^2 R^{n-2} + s_2^2 R^{n-2}) = \frac{R^{-2}(\rho_1 + \rho_2)}{2}$$

and

$$\begin{aligned} \int_{\mathbb{R}^n} |u_1 u_2|^\gamma &= R^n (s_1 s_2)^\gamma \|v\|_{2\gamma}^{2\gamma} = R^n \lambda^\gamma s_1^{2\gamma} \|v\|_{2\gamma}^{2\gamma} \\ &= R^{n(1-\gamma)} (\lambda \rho_1)^\gamma \|v\|_{2\gamma}^{2\gamma} \\ \int_{\mathbb{R}^n} G(u) &\leq c_0 \int_{\mathbb{R}^n} |u|^p + |u|^q = c_0 R^n (s_1^p (1 + \lambda^2)^{p/2} + s_1^q (1 + \lambda^2)^{q/2}) \\ &= c_0 R^{n(1-\frac{p}{2})} (1 + \lambda^2)^{p/2} + c_0 R^{n(1-\frac{q}{2})} (1 + \lambda^2)^{q/2}. \end{aligned}$$

The constant  $c_0$  above follows from (A2). From the definition of  $J$  and (A1) and (A2), there exists a positive constant  $c > 0$  such that

$$J(u) \leq c \left( R^{-2} - R^{n(1-\gamma)} + R^{n(1-\frac{p}{2})} + R^{n(1-\frac{q}{2})} \right).$$

For  $R$  large enough the right term of the inequality above is negative. Now we prove that the infimum of  $J$  is finite. We fix  $\rho$  as above. By the Hölder inequality and (A3), we have

$$\begin{aligned} (11) \quad J(u) &\geq \frac{1}{2} \sum_{j=1}^2 \|Du_j\|^2 - 2\beta (\|u_1\|_{2\gamma} \|u_2\|_{2\gamma})^\gamma \\ &\geq \frac{1}{2} \sum_{j=1}^2 (\|Du_j\|^2 - 2\beta \|u_j\|_{2\gamma}^{2\gamma}) \end{aligned}$$

From (1) and the interpolation inequality, we have

$$\begin{aligned}
 \|u_j\|_{2\gamma} &\leq \|u_j\|^{1-\frac{n}{2}+\frac{n}{2\gamma}} \|u_j\|_{2^*}^{\frac{n}{2}-\frac{n}{2\gamma}} \\
 (12) \quad &\leq S^{\frac{n}{2}-\frac{n}{2\gamma}} \|u_j\|^{1-\frac{n}{2}+\frac{n}{2\gamma}} \|Du_j\|^{\frac{n}{2}-\frac{n}{2\gamma}} \\
 &= S^{\frac{n}{2}-\frac{n}{2\gamma}} \rho_j^{\frac{1}{2}(1-\frac{n}{2}+\frac{n}{2\gamma})} \|Du_j\|^{\frac{n}{2}-\frac{n}{2\gamma}}.
 \end{aligned}$$

Then, there exists a constant  $c = c(\beta, \rho_j, S) > 0$  such that

$$\begin{aligned}
 J(u) &\geq c \sum_{j=1}^2 \|Du_j\|^2 - \|Du_j\|^{\frac{n}{2}-\frac{n}{2\gamma}} \\
 (13) \quad &= c \sum_{j=1}^2 X_j^2 - X_j^{\frac{n}{2}-\frac{n}{2\gamma}} =: g(X)
 \end{aligned}$$

where in the last equality  $\|Du_j\|_2$  has been replaced by  $X_j$ . By the hypotheses on  $\gamma$  in (A1),  $g$  is bounded from below. Thus,  $I_\rho$  is well-defined and negative.

(ii). Let  $(u_n)$  be a minimizing sequence on  $N_\rho$ . By definition,  $\|u_n^j\|_2^2 = \rho_j$  thus constant and bounded. Moreover, if a subsequence of  $\|Du_n^j\|_2$  diverges, for some  $j = 1, 2$ , then the right end of the first line in (13) will diverge positively, leading to a contradiction with  $I_\rho < 0$ .  $\square$

### 3. SOLUTIONS ON BOUNDED DOMAINS

Given  $R > 0$ , we denote with  $B_R$  the ball centered at the origin with radius  $R$ . We define the functional

$$(14) \quad J_R: H_0^1(B_R) \oplus H_0^1(B_R) \rightarrow \mathbb{R}$$

as the restriction of  $J$ . We look at the minimizers of  $J_R$  over the constraint

$$N_\rho(B_R) := \{u \in H_0^1(B_R) \oplus H_0^1(B_R) \mid \|u_j\|_{L^2(B_R)}^2 = \rho_j\}.$$

The assumptions on  $F$  are those stated in the previous section (even if some of them could be relaxed).

**Proposition 3.** *The functional  $J_R$  attains its infimum on  $N_\rho(B_R)$ . If  $J_R(u) = \inf_{N_\rho(B_R)} J_R$ , then  $u_j > 0$  or  $u_j < 0$  on  $B_R$ . Moreover, a minimum of  $J_R$  can be chosen to be positive, radially symmetric and decreasing, and of class  $C^{1,\alpha}(\overline{B})$ .*

*Proof.* In order to simplify the notation, we denote  $J_R$  with  $J$ ,  $B_R$  with  $B$  and  $N_\rho(B_R)$  with  $N$ . Let  $L$  be the Lagrangian associated to  $J$

$$\begin{aligned}
 L: B \times \mathbb{R}^2 \times \mathbb{R}^{2n} &\rightarrow \mathbb{R} \\
 (x, z, p) &\mapsto \frac{1}{2}|p|^2 - \beta|z_1 z_2|^\gamma + G(z)
 \end{aligned}$$

First, we observe that  $J$  is weakly lower semi-continuous. Let  $\lambda \in \mathbb{R}$  and  $2^* > r > 2\gamma$  be such that

$$L_\lambda(x, z, p) := L(x, z, p) + \lambda(|z|^2 + |z|^r) \geq 0$$

Such  $\lambda$  and  $r$  exist from the hypothesis on  $\gamma$  in (A1) and (A2). Because  $L$  is convex in  $p$ , also  $L_\lambda$  is convex in  $p$ . We denote with  $J_\lambda$  the functional associated to  $L_\lambda$ . Thus,  $J_\lambda$  is weakly lower semi-continuous by [23, Theorem 1.6, p. 9]. Given a weakly converging sequence  $u_n \rightharpoonup u$  in  $H$ , there exists a subsequence  $u_{n_k}$  such that

$$\begin{aligned} u_{n_k} &\rightarrow u \in L^2(B_R) \cap L^r(B_R) \\ \liminf_{n \rightarrow \infty} J(u_n) &= \lim_{k \rightarrow \infty} J(u_{n_k}) \end{aligned}$$

from the Rellich-Kondrakov theorem, [8, Théorème IX.16, p. 169]. We have

$$\begin{aligned} \liminf_{n \rightarrow \infty} J(u_n) &= \lim_{k \rightarrow \infty} J(u_{n_k}) \\ &= -\lambda (\|u_{n_k}\|^2 + \|u_{n_k}\|_r^r) + \lim_{k \rightarrow \infty} J_\lambda(u_{n_k}) \\ &= -\lambda (\|u\|^2 + \|u\|_r^r) + \liminf_{k \rightarrow \infty} J_\lambda(u_{n_k}) \\ &\geq -\lambda (\|u\|^2 + \|u\|_r^r) + J_\lambda(u) = J(u). \end{aligned}$$

Let  $u_n$  be a minimizing sequence for  $J$  over  $N$ . By (ii) of Lemma 1 such sequence is bounded. Thus, by the Rellich-Kondrakov theorem, we can suppose that  $u_n \rightarrow u$  in  $L^2(B)$  and weakly in  $H_0^1(B)$ . Then  $u \in N$ . Because  $J$  is weakly lower semi-continuous, we have

$$J(u) \leq \liminf_{n \rightarrow +\infty} J(u_n) = \inf_N J.$$

Hence  $u$  is a minimizer of  $J$ . We argue by contradiction and suppose that  $u$  vanishes at some point. Let

$$v_j := |u_j| \geq 0.$$

It is easy to check that  $v \in N$  and  $J(u) = J(v)$ . Then  $v$  is a weak solution of the

$$(15) \quad -\Delta v_j = \lambda_j v_j + \beta \gamma v_j^{\gamma-1} v_{\sigma(j)}^\gamma - D_j G(v)$$

for some  $\lambda_j \in \mathbb{R}$  and where  $\sigma(1) = 2$  and  $\sigma(2) = 1$ . By local regularity theory  $v_j \in H_{loc}^2(B) \cap C(\overline{B})$  and  $v_j$  vanishes at some point. We have

$$-\Delta v_j - \lambda_j v_j + D_j G(v) \geq 0.$$

By (A1, A2), we have a well defined function

$$A_j(x) = \begin{cases} \lambda_j - D_j G(v) v_j^{-1} & \text{if } v_j(x) \neq 0 \\ \lambda_j & \text{otherwise.} \end{cases}$$

Therefore

$$\Delta v_j + A_j^-(x) v_j \leq 0.$$

Because  $v_j$  is bounded on  $B$  and  $D_j G$  is continuous, we have  $A_j^- \in L^\infty(B)$ . Thus we can apply the strong maximum principle: if  $v_j$  vanishes in the interior of  $B$ , by [13, Theorem 3.5],  $v_j \equiv 0$ . Because this is not possible by the constraint condition, we obtain  $v_j > 0$ , a contradiction.

Now, given a positive minimizer  $u$ , we can take the decreasing rearrangement  $u^*$ . By [17, Eq. (4), p. 81]

$$u^* \in N_\rho(B).$$

Moreover, by [17, Lemma 7.17, p. 188], [16, Lemma 2.1] and assumption (A4), it follows that

$$J(u^*) \leq J(u).$$

In fact, due to the minimization property of  $u$ , the inequality is an equality, hence  $u^*$  is a minimizer. Because of the radial symmetry, we have  $u^* \in C^{1,\alpha}(\overline{B})$  for some  $\alpha \in (0, 1)$ .  $\square$

#### 4. THE SUB-ADDITIVITY PROPERTY OF $I$

The next Lemma is the one-dimensional case of [10, Proposition 1.4]. We include the proof because, due to the specificity of the case, we can state a more precise inequality. We use the notation  $u^*$  for the symmetric decreasing rearrangement.

**Lemma 2.** *Let  $u, v \in H^1(\mathbb{R})$  be two compactly supported, symmetric functions with respect to the origin such that*

$$\text{supp}(u) = [-c, c], \quad \text{supp}(v) = [-d, d]$$

*and  $\sup(u) \leq \sup(v)$ . Moreover,  $u$  and  $v$  are differentiable except on the boundary and*

$$(16) \quad tu'(t), tv'(t) < 0$$

*on the complementary of a finite subset. Let  $T$  be such that*

$$\text{supp}(u) \cap \text{supp}(v(\cdot - T)) = \emptyset.$$

*We define  $w(t) := u(t) + v(t - T)$ . Then*

$$(17) \quad \|w^{*'}\|^2 \leq \|w'\|^2 - \frac{3}{4}\|u'\|^2.$$

*Proof.* We set  $a := \sup(u)$  and  $b := \sup(v)$ . The functions

$$u: (0, c) \rightarrow (0, a), \quad v: (0, d) \rightarrow (0, b)$$

are invertible, because they are strictly decreasing, by (16). Let  $y_u$  and  $y_v$  be these inverses. Thus,

$$(18) \quad u(y_u(s)) = s \text{ on } (0, a), \quad v(y_v(s)) = s \text{ on } (0, b)$$

Because  $w^*$  is symmetric and decreasing, the level set  $\{w^* > s\}$  is an interval. We define its width by  $2z(s)$ . We have

$$(19) \quad 2z(s) = |\{w^* > s\}| = \begin{cases} 2y_u(s) + 2y_v(s) & \text{if } s \in (0, a) \\ 2y_v(s) & \text{if } s \in (a, b). \end{cases}$$

The second equality follows from the definition of decreasing rearrangement. Because  $y_u$  and  $y_v$  are strictly decreasing functions and differentiable everywhere, so is  $z$ . Moreover

$$(20) \quad w^*(z(s)) = s \text{ on } (0, b).$$

Taking the derivative with respect to  $s$  in (20) and in (18), we have

$$(21) \quad w^{*'}(z(s))z'(s) = 1, \quad u'(y_u(s))y'_u(s) = 1, \quad v'(y_v(s))y'_v(s) = 1.$$

Hence

$$(22) \quad \begin{aligned} \int_{\mathbb{R}} |w^{*'}|^2 dt &= 2 \int_0^{c+d} |w^{*'}|^2 \\ &= -2 \int_0^b |w^{*'}(z(s))|^2 z'(s) ds = -2 \int_0^b (z'(s))^{-1} ds \\ &= -2 \int_0^a (y'_u(s) + y'_v(s))^{-1} - 2 \int_a^b (y'_v(s))^{-1} ds. \end{aligned}$$

The second equality follows from a change of variable and (21). The fourth equality follows from (19). We use the inequality

$$2(x + y)^{-1} \leq x^{-1} + y^{-1} - \max\{x^{-1}, y^{-1}\}.$$

Thus, the last term of (22) is bounded from above by

$$\begin{aligned} & - \int_0^a (y'_u(s))^{-1} + (y'_v(s))^{-1} ds \\ & + \int_0^a \max\{y'_u(s)^{-1}, y'_v(s)^{-1}\} ds - 2 \int_a^b (y'_v(s))^{-1} ds \end{aligned}$$

using the estimate  $2 \max\{t, s\} \geq t + s$ , the last term is bounded by

$$\begin{aligned} & - \frac{1}{2} \int_0^a (y'_u(s))^{-1} ds - \frac{1}{2} \int_0^a (y'_v(s))^{-1} ds - 2 \int_a^b (y'_v(s))^{-1} ds \\ & \leq - \frac{1}{2} \int_0^a (y'_u(s))^{-1} ds - 2 \int_0^b (y'_v(s))^{-1} ds \\ & = \frac{1}{4} \cdot \left( -2 \int_0^a (y'_u(s))^{-1} \right) + \left( -2 \int_0^b (y'_v(s))^{-1} \right) ds. \end{aligned}$$

From a change of variable and (21) it follows

$$\|u'\|^2 = -2 \int_0^a (y'_u(s))^{-1} ds, \quad \|v'\|^2 = -2 \int_0^b (y'_v(s))^{-1} ds.$$

Thus, from (22), we obtain

$$\|w^{*'}\|^2 \leq \frac{1}{4}\|u'\|^2 + \|v'\|^2 = \|w'\|^2 - \frac{3}{4}\|u'\|^2.$$

□

**Proposition 4.** *Let  $\rho, \tau$  be such that  $\rho_i \geq \tau_i > 0$  and  $\tau \neq \rho$ . Then,*

$$I_\rho < I_\tau + I_{\rho-\tau}$$

*that is,  $f$  is sub-additive.*

*Proof.* Let us define  $\sigma := \rho - \tau$  and let

$$(23) \quad u_n \in N_\tau, \quad v_n \in N_\sigma$$

be minimizing sequences of  $J$  over  $N_\tau$  and  $N_\sigma$ , respectively. Because  $J$  is continuous by (a) of Proposition 2, we can suppose that all the functions above have compact support. Thus, there exists a ball  $B_n \subset \mathbb{R}^n$  such that

$$\text{supp}(u_n) \cup \text{supp}(v_n) \subseteq B_n.$$

By Proposition 3, we can replace  $u_n$  and  $v_n$  with the corresponding minimizers of  $J$  over  $N_\tau(B_n)$  and  $N_\sigma(B_n)$ . By the same proposition, these can be chosen to be positive in the interior of  $B_n$ , radially symmetric and  $C^1(\overline{B_n})$ . Using the Stone-Weierstrass theorem, we can suppose that

$$u_n^j(x) := p_n^j(|x|)$$

where  $p_n$  is a polynomial. Hence,

$$(24) \quad u_n^j(x', \cdot)'(t) = \frac{tp_n^{j'}(|(t, x')|)}{\sqrt{t^2 + |x'|^2}}.$$

Thus,  $u_n^j(x', \cdot)'$  vanishes in a finite number of points, for every  $x' \in \mathbb{R}^{n-1}$ . Because  $u_n^j$  and  $v_n^j$  have compact support, there exists a real sequence  $(t_n)$  such that the two functions

$$u_n^j, v_n^j(\cdot + t_n e_n)$$

have disjoint support, where  $e_n = (0, \dots, 0, 1)$ . Then, we can apply Lemma 2 to

$$(25) \quad w_n := u_n + v_n(\cdot + t_n e_n) \in N_\rho$$

$$(26) \quad J(w_n) = J(u_n) + J(v_n).$$

We denote with  $w_n^{*e_n}$  the Steiner symmetrization of  $w_n$  with respect to  $e_n$ . The Steiner symmetrization has the same properties of the decreasing rearrangement we used in Section 3. That is,

$$\|w_n^{*e_n}\| = \|w_n^j\|, \quad J(w_n^{*e_n}) \leq J(w_n).$$

Given  $x' \in \mathbb{R}^{n-1}$ , the relation between the Steiner and the decreasing rearrangement gives

$$D_n w_n^{*e_n}(x', t) = w_n^j(x', \cdot)'(t)$$

Then, we can write

$$\begin{aligned}
 (27) \quad & \int_{\mathbb{R}^n} |D_n w_n^{j*e_n}|^2 = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} |w_n^{j*}(x', \cdot)'(t)|^2 dt dx' \\
 & = \int_{U_n^j} \int_{\mathbb{R}} |w_n^{j*}(x', \cdot)'(t)|^2 dt dx' + \int_{V_n^j} \int_{\mathbb{R}} |w_n^{j*}(x', \cdot)'(t)|^2 dt dx' =: A_1^j + A_2^j
 \end{aligned}$$

where

$$\begin{aligned}
 U_n^j &= \{x' \in \mathbb{R}^{n-1} \mid \sup_{\mathbb{R}} u_n^j(x', \cdot) \leq \sup_{\mathbb{R}} v_n^j(x', \cdot)\} \\
 V_n^j &= \{x' \in \mathbb{R}^{n-1} \mid \sup_{\mathbb{R}} v_n^j(x', \cdot) < \sup_{\mathbb{R}} u_n^j(x', \cdot)\}.
 \end{aligned}$$

For every  $x' \in \mathbb{R}^{n-1}$ ,  $u_n^j(x', \cdot)$  and  $v_n^j(x', \cdot + te_n)$  satisfy the hypothesis of Lemma 2, by Proposition 3 and (24). Thus,

$$\begin{aligned}
 A_1^j &\leq \int_{U_n^j} \left( \|w_n(x', \cdot)'\|_{L^2(\mathbb{R})}^2 - \frac{3}{4} \|u_n^j(x', \cdot)'\|_{L^2(\mathbb{R})}^2 \right) dx' \\
 A_2^j &\leq \int_{V_n^j} \left( \|w_n^j(x', \cdot)'\|_{L^2(\mathbb{R})}^2 - \frac{3}{4} \|v_n^j(x', \cdot)'\|_{L^2(\mathbb{R})}^2 \right) dx'.
 \end{aligned}$$

In the last equalities we used

$$\|v_n^j(x', \cdot)\|_{L^2(\mathbb{R})} = \|v_n^j(x', \cdot + t_n e_n)\|_{L^2(\mathbb{R})}$$

and the fact that the definition of  $U_n^j$  and  $V_n^j$  do not change if we shift the last coordinate. Thus,

$$\begin{aligned}
 (28) \quad & A_1^j + A_2^j \leq \|D_n w_n^j\|^2 \\
 & - \frac{3}{4} \left( \|D_n u_n^j\|_{L^2(U_n^j \times \mathbb{R})}^2 + \|D_n v_n^j\|_{L^2(V_n^j \times \mathbb{R})}^2 \right).
 \end{aligned}$$

Because  $u_n$  is radially symmetric, then

$$D_i u_n^j(x_1, \dots, x_i, \dots, x_n) = D_n u_n^j(x_1, \dots, x_n, \dots, x_i)$$

for every  $1 \leq i \leq n$ . Then, for every measurable subset  $A \subset \mathbb{R}^n$ , by the equality above

$$\|D_n u_n^j\|_{L^2(A)} = \|D_i u_n^j\|_{L^2(A_i)}$$

where  $A_i$  is the set obtained by  $A$  through the permutation of the coordinates  $i$  and  $n$ . Moreover,

$$\|D_i u_n^j\|_{L^2(A_i)}^2 = \|D_i u_n^j\|_{L^2(A)}^2.$$

Because  $D_i u_n^j$  is radially symmetric. In conclusion (28) can be written as

$$\begin{aligned}
 (29) \quad & n(A_1^j + A_2^j) \leq \|D w_n^j\|^2 \\
 & - \frac{3}{4} \left( \|D u_n^j\|_{L^2(U_n^j \times \mathbb{R})}^2 + \|D v_n^j\|_{L^2(V_n^j \times \mathbb{R})}^2 \right)
 \end{aligned}$$



We define

$$(30) \quad d_n^j = \|Du_n^j\|_{L^2(U_n^j \times \mathbb{R})}^2 + \|Dv_n^j\|_{L^2(V_n^j \times \mathbb{R})}^2.$$

We will prove that a subsequence of  $(d_n^j)$  is bounded from below. Because  $u_n$  and  $v_n$  are minimizing sequences, from (11) and (i) of Lemma 1

$$(31) \quad \|u_n^j\|_{2\gamma}, \|v_n^j\|_{2\gamma} \geq c = c(\rho, \tau) > 0.$$

Because they are radially symmetric, by [7, Theorem A.I'], up to extract a subsequence we can suppose that

$$u_n^j \rightarrow u_j, v_n^j \rightarrow v_j \text{ in } L^{2\gamma}(\mathbb{R}^n)$$

and  $u_j, v_j \not\equiv 0$  by (31). Up to extract a subsequence, we can suppose that the convergence is pointwise a.e. Thus, there are points  $x_j, y_j$  other than 0 such that

$$(32) \quad u_j(x_j), v_j(y_j) \geq h > 0.$$

Then for  $n \geq n_1$ ,

$$(33) \quad u_n^j(x_j), v_n^j(y_j) \geq h/2.$$

We define  $R = \min \{|x_j|, |y_j| \mid j = 1, 2\}$ . Because  $u_n^j$  and  $v_n^j$  are radially decreasing, for every  $n \geq n_1$  and  $x \in B(0, r)$ , we have

$$(34) \quad u_n^j(x) \geq h/2, \quad v_n^j(x) \geq h/2.$$

By applying the second inequality of (12) to the domains

$$(U_n^j \times \mathbb{R}) \cap B_r, \quad (V_n^j \times \mathbb{R}) \cap B_r,$$

because  $u_n \in N_\tau$  and  $v_n \in N_\sigma$ , and  $\rho_j \geq \sigma_j, \tau_j$ , from (34), there exists  $c_j = c_j(\rho_j, S) > 0$  such that

$$\begin{aligned} \|Du_n^j\|_{L^2(U_n^j \times \mathbb{R})} &\geq \|Du_n^j\|_{L^2((U_n^j \times \mathbb{R}) \cap B_r)} \\ &\geq c \left( \frac{h|(U_n^j \times \mathbb{R}) \cap B_r|^{\frac{1}{2\gamma}}}{2} \right)^{\frac{2\gamma}{n(\gamma-1)}} \end{aligned}$$

and

$$\begin{aligned} \|Dv_n^j\|_{L^2(V_n^j \times \mathbb{R})} &\geq \|Dv_n^j\|_{L^2((V_n^j \times \mathbb{R}) \cap B_r)} \\ &\geq c \left( \frac{h|(V_n^j \times \mathbb{R}) \cap B_r|^{\frac{1}{2\gamma}}}{2} \right)^{\frac{2\gamma}{n(\gamma-1)}} \end{aligned}$$

Because  $U_n^j \cap B_R$  is the complementary of  $V_n^j \cap B_R$  in  $B_R$ , for every  $j = 1, 2$ , at least one of the two quantities

$$\delta_j := \liminf_{n \rightarrow \infty} \frac{|(U_n^j \times \mathbb{R}) \cap B_R|}{|B_R|}, \quad 1 - \delta_j$$

is not smaller than 1/2. Then,

$$d_n^j \geq 2^{-\frac{2(2\gamma+1)}{n(\gamma-1)}} h^{\frac{4\gamma}{n(\gamma-1)}} |B_R|^{\frac{1}{n(\gamma-1)}} c_j =: d_j$$

for  $j = 1, 2$ . Then, from (27,29) and the inequality above we obtain

$$(35) \quad n \int_{\mathbb{R}^n} |D_n w_n^{j*e_n}|^2 \leq \|Dw_n^j\|^2 - \frac{3d_n^j}{4} \leq \|Dw_n^j\|^2 - \frac{3d_j}{4}.$$

Finally, we consider the decreasing rearrangement of  $w_n^{*e_n}$  (which may not be radially symmetric). By [16, Lemma 11], when  $n = 1$ , we have

$$\begin{aligned} \|D_n w_n^{j*e_n*}\|^2 &= \int_{\mathbb{R}^{n-1}} \|w_n^{j*e_n*}(x', \cdot)'\|_{L^2(\mathbb{R})}^2 dx' \\ &\leq \int_{\mathbb{R}^{n-1}} \|w_n^{j*e_n}(x', \cdot)'\|_{L^2(\mathbb{R})}^2 dx' = \|D_n w_n^{j*e_n}\|^2. \end{aligned}$$

From (35), we can write

$$n \int_{\mathbb{R}^n} |D_n w_n^{j*e_n*}|^2 \leq \|Dw_n^j\|^2 + \|Dv_n^j\|^2 - \frac{3d_j}{4}.$$

Because  $w_n^{j*e_n*}$  is radially symmetric

$$\int_{\mathbb{R}^n} |Dw_n^{j*e_n*}|^2 \leq \|Dw_n^j\|^2 + \|Dv_n^j\|^2 - \frac{3d_j}{4}.$$

Hence,

$$I_\rho \leq J(w_n^{*e_n*}) \leq J(u_n) + J(v_n) - \frac{3d}{8}$$

where  $d := d_1 + d_2$ . Taking the limit as  $n \rightarrow \infty$ , we obtain

$$I_\rho \leq I_\tau + I_\sigma - D$$

where  $D := 3d/8 > 0$ . □

## 5. CONCENTRATION OF MINIMIZING SEQUENCES IN $\mathbb{R}^n$

In this section we establish the existence of a minimizer for  $J$  on  $\mathbb{R}^n$  and a concentration property of minimizing sequences on the constraint  $N_\rho$ . We start with the following

**Lemma 3.** *Let  $(u_n)_{n \geq 1}$  be a bounded sequence in  $H$  such that*

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^n} |u_n^1 u_n^2|^\gamma > 0$$

*where  $1 < \gamma < 2^*/2$ . Then, there exists a sequence  $(y_n) \subset \mathbb{R}^n$  such that*

$$u_n^j(\cdot - y_n) \rightharpoonup u_j, \quad u_1 u_2 \not\equiv 0.$$

*Proof.* Let  $w_n = u_n^1 u_n^2$ . From the Hölder inequality and (2), it follows that

$$Dw_n \in L^{n/n-1}.$$

We apply [18, Lemma I.1, p. 231] with  $q = 1$  and  $p = n/n - 1$ . Hence, either there exists  $R > 0$  and a sequence  $(y_n)$  such that

$$(36) \quad \liminf_{n \rightarrow \infty} \int_{B(-y_n, R)} |w_n| > 0$$

or

$$(37) \quad w_n \rightarrow 0 \text{ in } L^\alpha, \text{ for every } \alpha \in (1, n/n-2).$$

The latter cannot happen because, by (A1)

$$\gamma < \frac{2^*}{2} = \frac{n}{n-2}.$$

So, (37) would contradict the hypothesis of the Lemma. Hence (36) holds. By changing the variable of integration in (36) and letting

$$v_n^j = u_n^j(\cdot - y_n)$$

we obtain

$$(38) \quad \liminf_{n \rightarrow \infty} \int_{B_R} v_n^1 v_n^2 > 0.$$

Since  $v_n^j$  are bounded in  $H^1$ , we can suppose that they converge weakly to some limits  $u_1$  and  $u_2$ , respectively. By the Rellich-Kondrakhov theorem, we can suppose that such convergence is strong in  $L^2(B_R)$ . Thus, in (38) we can take the limit in the integrand and obtain

$$\int_{B_R} u_1 u_2 > 0$$

which implies  $u_1 u_2 \not\equiv 0$ . And  $u_n^j(\cdot - y_n) \rightharpoonup u_j$ .  $\square$

**Theorem 1.** *Let  $(u_n)_{n \geq 1}$  be a minimizing sequence for  $J$  over  $N_\rho$ . Then, there exists  $u \in N_\rho$  and a sequence  $(y_n)_{n \geq 1}$  such that*

$$u_n = u(\cdot + y_n) + o(1) \text{ in } H$$

$$J(u) = \inf_{N_\rho} J.$$

*Proof.* By (i) and (ii) of Lemma 1  $I_\rho < 0$  and the sequence  $(u_n)_{n \geq 1}$  is bounded. Because  $G \geq 0$  the sequence  $(u_n)$  fulfils the hypothesis of Lemma 3, once  $\gamma < n/(n-2)$  holds. This, in turn, follows from (A1) and

$$1 + \frac{2}{n} < \frac{n}{n-2}.$$

Then, we consider the sequence  $(y_n)_{n \geq 1}$  and  $u \in H$  given by the Lemma 3. We define

$$v_n := u_n(\cdot - y_n) - u, \quad \tau := (\|u_1\|_{L^2}^2, \|u_2\|_{L^2}^2).$$

We have  $\tau_i \leq \rho_i$ , by the weak lower semi-continuity property of the  $L^2$ -norm. Suppose that  $\tau \neq \rho$ . By (b) of Proposition 2, up to extract a subsequence, we have

$$(39) \quad J(v_n) = J(u_n(\cdot + y_n)) - J(u) + o(1).$$

By a change of variable, the first term of the right member equals  $J(u_n)$  which converges to  $I_\rho$ . Hence, by Lemma 4

$$I_{\rho-\tau} \leq I_\rho - I_\tau < I_{\rho-\tau} - D$$

for some constant  $D > 0$ . Thus  $\tau = \rho$  and  $u \in N_\rho$ . Moreover,

$$u_n(\cdot - y_n) - u \rightarrow 0 \text{ in } L^2(\mathbb{R}^n, \mathbb{R}^2).$$

By the weak lower semi-continuity property of  $J$ ,  $J(u_n) \rightarrow J(u)$ . This completes the first and the third statement of the Theorem. In order to complete the proof, we only need to show that the convergence above holds in  $H$  as well. Because

$$(40) \quad (u_n(\cdot + y_n))_{n \geq 1}$$

is a minimizing sequence, by Ekeland's Theorem [23, Theorem 5.1, p. 51] there exists a sequence  $w_n$  such that

$$\|w_n - u_n(\cdot + y_n)\|_{H^1} \rightarrow 0.$$

and  $(w_n)_{n \geq 1}$  is Palais-Smale. That is, there are  $\lambda_1, \lambda_2 \in \mathbb{R}$  such that,

$$X_n := DJ(w_n) - (\lambda_1 w_n^1, \lambda_2 w_n^2) \rightarrow 0.$$

Then

$$(41) \quad \sum_{j=1}^2 (X_n^j, w_n^j - w_m^j) = \sum_{j=1}^2 \|Dw_n^j - Dw_m^j\|^2 - \lambda_j \|w_n^j - w_m^j\|^2 \\ + \int_{\mathbb{R}^n} (D_j F(w_n) - D_j F(w_m))(w_n^j - w_m^j).$$

We look at the third summand of the right-end of the equality above. We have

$$D_1 F(w_n) = -\gamma \beta w_n^{1, \gamma-1} w_n^{2, \gamma} + D_1 G(w_n) \\ D_2 F(w_n) = -\gamma \beta w_n^{2, \gamma-1} w_n^{1, \gamma} + D_2 G(w_n).$$

By (A2), we have

$$(42) \quad \left| \int_{\mathbb{R}^n} D_j G(w_n)(w_n^j - w_m^j) \right| \leq c_1 \int_{\mathbb{R}^n} |w_n|^{p-1} |w_n^j - w_m^j| \\ + c_1 \int_{\mathbb{R}^n} |w_n|^{q-1} |w_n^j - w_m^j|.$$

We apply the Hölder inequality to each of the integrands with pairs  $(p', p)$  and  $(q', q)$  respectively. From inequality (12) where  $2\gamma$  is replaced by  $p$  and  $q$ , it follows

$$\int_{\mathbb{R}^n} |w_n|^{p-1} |w_n^j - w_m^j| \leq \|w_n\|_p^{p-1} \|w_n^j - w_m^j\|_p \\ \leq c \|w_n^j - w_m^j\|^\theta$$

for some  $c > 0$  and  $\theta \in (0, 1)$ . A similar inequality holds for  $q$ . Then the left member in (42) converges to zero for  $i = 1, 2$ . We use the Hölder inequality to estimate the term

$$(43) \quad \gamma \beta \int_{\mathbb{R}^n} w_n^{1, \gamma-1} w_n^{2, \gamma} |w_n^j - w_m^j|$$

as well. In order to do so, we need to find a triple of real numbers  $(r_1, r_2, r_3)$  such that

$$r_1(\gamma - 1), r_2\gamma \in [2, 2^*], r_3 \in [2, 2^*)$$

$$\sum_{i=1}^3 \frac{1}{r_i} = 1, \quad r_i \geq 1.$$

We briefly check that we can achieve such a triple. The first line can be written as

$$r_1^{-1} \in [(\gamma - 1)/2^*, (\gamma - 1)/2], \quad r_2^{-1} \in [\gamma/2^*, \gamma/2]$$

$$1 - r_1^{-1} - r_2^{-1} \in (1/2^*, 1/2].$$

From (A1),  $2/\gamma - 1 > 2/\gamma > 1$  for  $n \geq 3$ . Thus, the requirement  $r_i \geq 1$  is included in the first of the two lines above. Then, it is enough to check that

$$[1 - (2\gamma - 1)/2, 1 - (2\gamma - 1)/2^*] \cap (1/2^*, 1/2] \neq \emptyset.$$

In fact, the intersection above is empty if and only if either

$$1 - \frac{2\gamma - 1}{2^*} \leq \frac{1}{2^*} \Rightarrow \gamma \geq \frac{n}{n - 2} > 1 + \frac{2}{n}$$

or

$$\frac{1}{2} < 1 - \frac{2\gamma - 1}{2} \Rightarrow \gamma < 1.$$

Both of them contradict (A1). Hence (43) can be estimated from above by

$$c\|w_n^j - w_m^j\|^\theta$$

for some  $c > 0$  and  $\theta \in (0, 1)$ . Thus, from (41)  $(Dw_n^j)_{n \geq 1}$  is a Cauchy sequence and

$$w_n^j \rightarrow u_j, \quad Dw_n^j \rightarrow f_j$$

for some  $f_j \in L^2(\mathbb{R}^n)$ . Hence  $f_j = Du_j$  and  $w_n \rightarrow u$  in  $H$ . By (40)

$$u_n(\cdot - y_n) - u \rightarrow 0 \text{ in } H$$

By a change of variable, the second statement of the Theorem also follows.  $\square$

Let  $m_1, m_2$  such that

$$(A5) \quad F(u) + \frac{1}{2}(m_1^2 u_1^2 + m_2^2 u_2^2) \geq 0.$$

Under this assumption we prove Theorem A.

*Proof of Theorem A.* From [12, Lemma 1] minimizing sequences of  $E$  over  $M_C$  are bounded in  $H \oplus \mathbb{R}^2$ . Thus, up to extract a subsequence

$$\|u_n^j\|^2 \rightarrow \rho_j, \quad \omega_n \rightarrow \omega.$$

As in *Step II* of the proof [2, Lemma 2.7] it can be shown that

$$v_n^i = \frac{u_n^i \sqrt{\rho_i}}{\|u_n^i\|}$$

is a minimizing sequence for  $J$  over  $N_\rho$ . Then, by Theorem 1, there exists a sequence  $(y_n)_{n \geq 1} \subset \mathbb{R}^n$  such that

$$v_n(\cdot + y_n) \rightarrow u \text{ in } H$$

for some  $u \in H$ . Then,  $(u, \omega) \in M_C$  is a minimum of  $E$  over  $M_C$ .  $\square$

## 6. LYAPUNOV FUNCTIONS

In the space  $X := H^1(\mathbb{R}^n, \mathbb{C}^2) \oplus L^2(\mathbb{R}^n, \mathbb{C}^2)$  we denote with  $d$  the metric induced by the scalar product

$$\langle \Phi, \Psi \rangle_{\mathbb{R}} := \operatorname{Re} \langle \Phi, \Psi \rangle_{\mathbb{C}} = \operatorname{Re} \sum_{j=1}^2 \int_{\mathbb{R}^n} \phi_j \bar{\psi}_j + \phi_t^j \bar{\psi}_t^j.$$

**Lemma 4.** *Let  $\phi \in H^1(\mathbb{R}^n, \mathbb{R}^m)$ . Then,  $|\phi| \in H^1(\mathbb{R}^n)$  and the inequality*

$$\|D\phi\| \geq \|D|\phi|\|.$$

*holds. If the equality holds and  $|\phi| > 0$  everywhere, then there exists  $\lambda \in \mathbb{R}^m$  such that  $|\lambda| = 1$  and*

$$\phi(x) = \lambda |\phi(x)|.$$

*Proof.* Clearly,  $|\phi| \in L^2(\mathbb{R}^n, \mathbb{R})$ . By applying [13, Theorem 7.4, p. 150] and [13, Theorem 7.8, p. 153] with  $f(\varphi) = |\varphi|$ , we obtain

$$D_i |\phi| = \begin{cases} \frac{\langle \phi, D_i \phi \rangle}{|\phi|} & \text{if } \phi \neq 0 \\ 0 & \text{if } \phi = 0, \end{cases}$$

By the Schwarz inequality

$$\begin{aligned} (44) \quad |D|\phi||^2 &= \sum_{i=1}^n |D_i |\phi||^2 = \frac{1}{|\phi|^2} \sum_{i=1}^n |\langle \phi, D_i \phi \rangle|^2 \\ &\leq \sum_{i=1}^n |D_i \phi|^2 = |D\phi|^2 \end{aligned}$$

if  $\phi \neq 0$ . When  $\phi = 0$ , the same inequality follows easily. Therefore,  $D|\phi| \in L^2(\mathbb{R}^n, \mathbb{R}^m)$  and, by integration, the first part of our statement is proved. Now, we suppose that the

$$\|D|\phi|\|^2 = \|D\phi\|^2.$$

Because  $\phi \neq 0$ , by (44) we obtain

$$|\phi| |D_i |\phi|| = |\langle \phi, D_i \phi \rangle|.$$

Because  $\phi \neq 0$ , there exists  $\mu_i: \mathbb{R}^n \rightarrow \mathbb{R}$ , such that

$$D_i \phi = \mu_i \phi.$$

Thus, for every  $1 \leq j \leq m$ , we have

$$D_i \phi_j = \mu_i \phi_j$$

We claim that each of the functions

$$\lambda_j: \mathbb{R}^n \rightarrow \mathbb{R}, \quad x \mapsto \frac{\phi_j(x)}{|\phi(x)|}$$

are constants. First, we notice that  $\lambda_j \in H_{loc}^1(\mathbb{R}^n)$ . In fact  $\lambda_j \in L^\infty(\mathbb{R}^n)$  and

$$D_i \lambda_j = \frac{D_i \phi_j |\phi|^2 - \phi_j \langle \phi, D_i \phi \rangle}{|\phi|^2}$$

hence,  $|D_i \lambda_j| \leq 2|D\phi| \in L^2(\mathbb{R}^n)$ . Moreover,

$$\sum_{h=1}^m D_i \phi_j \phi_h^2 - \phi_j \phi_h D_i \phi_h = \mu_i \sum_h \phi_j \phi_h^2 - \phi_j \phi_h^2 = 0.$$

Then  $\lambda_j$  is constant. Thus,

$$\phi_j = \lambda_j |\phi| \text{ on } \mathbb{R}^n.$$

We conclude the proof by choosing  $\lambda = (\lambda_1, \dots, \lambda_m)$ .  $\square$

A similar result of the Lemma above is known in [17, Theorem 7.8] under the more restrictive hypothesis that  $|\phi_k| > 0$  on  $\mathbb{R}^n$  for some  $1 \leq k \leq m$ .

*Proof of Theorem B.* Given  $\Phi \in \Gamma_C$ , there are  $\lambda_1, \lambda_2 \in \mathbb{C}$  and  $y \in \mathbb{R}^n$  such that  $|\lambda_j| = 1$  and

$$\Phi = (\lambda u(\cdot + y), -i\omega \lambda u(\cdot + y)).$$

Then

$$\mathbf{E}(\Phi) = E(u, \omega) = m_C, \quad \mathbf{C}_j(\Phi) = \omega_j \|u_j\|^2 = C_j.$$

Because  $\mathbf{E}$  and  $\mathbf{C}_j$  are continuous, if  $d(\Phi_n, \Gamma_C) \rightarrow 0$ , then

$$(45) \quad \mathbf{E}(\Phi_n) \rightarrow m_C, \quad \mathbf{C}_j(\Phi_n) \rightarrow C_j.$$

We prove the converse and suppose that (45) holds. Because  $m_C > 0$ , we can suppose that  $\phi_n^j \not\equiv 0$  for every  $n$ . From the Schwarz inequality, we obtain

$$(46) \quad \frac{\mathbf{C}_j(\phi_n, \phi_n^t)}{\|\phi_n^j\|} \leq \|\phi_n^{t,j}\|$$

By Lemma 4 and (46)

$$\begin{aligned}
 \mathbf{E}(\phi_n, \phi_n^t) &= \frac{1}{2} \int_{\mathbb{R}^n} |D\phi_n|^2 + |\phi_n^t|^2 + 2V(\phi_n) \\
 (47) \quad &\geq \frac{1}{2} \int_{\mathbb{R}^n} |D|\phi_n||^2 + 2V(|\phi_n^1|, |\phi_n^2|) + \frac{1}{2} \sum_{i=1}^2 \frac{\mathbf{C}_j(\phi_n, \phi_n^t)^2}{\|\phi_n^j\|^2}.
 \end{aligned}$$

We define

$$(48) \quad \omega_n^j = \frac{C_n^j}{\|\phi_n^j\|^2}, \quad u_n^j = |\phi_n^j|.$$

because of  $\mathbf{C}_j(\Phi_n) \rightarrow C_j$ , the sequence  $\omega_n^j$  will eventually become positive for  $n$  large enough. Then (47) implies

$$(49) \quad \mathbf{E}(\phi_n, \phi_n^t) \geq E(u_n, \omega_n) \geq m_C.$$

Taking the limit as  $n \rightarrow \infty$ , the first of (45) implies that  $(u_n, \omega_n)$  is a minimizing sequence for  $E$  over  $M_C$ . By Theorem A, there exists  $(u, \omega) \in M_C$  and a subsequence  $(y_n) \subset \mathbb{R}^n$  such that

$$(50) \quad |\phi_n| = u(\cdot + y_n) + o(1), \quad \omega_n = \omega + o(1).$$

We set

$$\psi_n := \phi_n(\cdot - y_n), \quad \psi_n^t := \phi_n^t(\cdot - y_n).$$

By a change of variable, we have

$$(51) \quad \mathbf{E}(\psi_n, \psi_n^t) = \mathbf{E}(\phi_n, \phi_n^t), \quad \mathbf{C}_j(\psi_n, \psi_n^t) = \mathbf{C}_j(\phi_n, \phi_n^t).$$

Up to extract a subsequence, we can suppose that there exists  $(\psi, \psi_t) \in X$  such that

$$(52) \quad \psi_n \rightharpoonup \psi \text{ in } H^1(\mathbb{R}^n, \mathbb{C}^2), \quad \psi_n^t \rightharpoonup \psi_t \text{ in } L^2(\mathbb{R}^n, \mathbb{C}^2).$$

By the weak lower semi-continuity of the norm, the strong convergence of  $|\psi_n|$  and Lemma 4, we have

$$\begin{aligned}
 \mathbf{E}(\psi_n, \psi_n^t) &= \frac{1}{2} \int_{\mathbb{R}^n} |D\psi_n|^2 + |\psi_n^t|^2 + 2V(\psi_n) \\
 &\geq \frac{1}{2} \int_{\mathbb{R}^n} |D\psi|^2 + |\psi_t|^2 + 2V(\psi) \\
 &\geq \frac{1}{2} \int_{\mathbb{R}^n} |D|\psi||^2 + 2V(\psi) + \frac{1}{2} \sum_{i=1}^2 \frac{\mathbf{C}_j(\psi, \psi_t)^2}{\|\psi_j\|^2} \geq m_C.
 \end{aligned}$$

Taking the limit as  $n \rightarrow \infty$ , by (51) and the first of (45), from the first inequality above, we obtain

$$(53) \quad \lim_{n \rightarrow \infty} \|\psi_n^t\| = \|\psi_t\|, \quad \lim_{n \rightarrow \infty} \|D\psi_n\| = \|D\psi\|.$$

From the second inequality we obtain

$$(54) \quad \int_{\mathbb{R}^n} |D\psi_j|^2 = \int_{\mathbb{R}^n} |D|\psi_j||^2, \quad \frac{\mathbf{C}_j(\psi, \psi_t)}{\|\psi_j\|} = \|\psi_t^j\|.$$



We show that  $\psi \equiv u$  almost everywhere. In fact, for every  $R > 0$ , from the first of (52), we have

$$\psi_n^j \rightharpoonup \psi_j \text{ in } L^2(B_R, \mathbb{C}).$$

Because  $\psi_n^j$  is bounded in  $H^1(B_R, \mathbb{C})$ , up to extract a subsequence, we can suppose that the two limits

$$\psi_n^j \rightarrow g_j \text{ in } L^2(B_R, \mathbb{C}), \quad |\psi_n^j| \rightarrow |g_j| \text{ in } L^2(B_R).$$

hold. By the uniqueness of the weak limit  $g_j = \psi_j$ . From (50),

$$|g_j| = u_j.$$

Then  $u_j \equiv |\psi_j|$  on  $B_R$ . Taking the limit as  $R \rightarrow \infty$ , we obtain

$$(55) \quad u = |\psi|.$$

Thus,  $\|\psi_n^j\| \rightarrow \|\psi_j\|$ . From the first of (52), we obtain

$$\psi_n \rightarrow \psi \text{ in } L^2(\mathbb{R}^n, \mathbb{C}).$$

Because  $(u, \omega)$  is a minimizer of  $E$  over  $M_C$ , then  $u_j > 0$ . This can be achieved with the argument used in Proposition 3. Then,  $|\psi_j| > 0$ . By (54) and Lemma 4, there are  $\lambda_j \in \mathbb{C}$  such that  $|\lambda_j| = 1$  and

$$\psi_j = \lambda_j |\psi_j| = u_j.$$

The second limit in (53) and the first in (52) give

$$D\psi_n^j \rightarrow D\psi_j.$$

Therefore

$$(56) \quad \psi_n^j \rightarrow \lambda_j u_j \text{ in } H^1(\mathbb{R}^n, \mathbb{C}).$$

The second equality in (54) can be written as follows

$$\operatorname{Re} \int_{\mathbb{R}^n} \overline{-i\psi_j} \cdot \psi_t^j = \|\psi_t^j\| \|\psi_j\|.$$

Thus, we have an equality between the scalar product and the product of norms. Then, there are  $\tilde{\omega}_j \in \mathbb{R}$  such that

$$(57) \quad \psi_t^j = -i\tilde{\omega}_j \psi_j.$$

Taking the limit in the first equality of (48), we obtain

$$\omega_j = \frac{C_j}{\|\psi_j\|^2}.$$

From (57) and (54), we also obtain  $\tilde{\omega}_j = \omega_j$ . Hence

$$\psi_j^t = -i\omega_j \psi_j.$$

By the second limit in (52) and the first of (53)

$$(58) \quad \psi_n^{j,t} \rightarrow \psi_t^j = -i\omega_j \lambda_j u_j \text{ in } L^2(\mathbb{R}^n, \mathbb{C}).$$

Thus, (56) and (58) give

$$d((\psi_n, \psi_n^t), \Gamma_C) \rightarrow 0$$

whence  $d((\phi_n, \phi_n^t), \Gamma_C) \rightarrow 0$ .  $\square$

The theorem can be restated by saying that

$$\mathbf{V}: X \rightarrow \mathbb{R}$$

$$(\phi, \phi_t) \mapsto (\mathbf{E}(\phi, \phi_t) - m_C)^2 + \sum_{j=1}^2 (\mathbf{C}_j(\phi, \phi_t) - C_j)^2$$

is a Lyapunov function for  $\Gamma_C$ , that is,  $d(\Phi_n, \Gamma_C) \rightarrow 0$  if and only  $\mathbf{V}(\Phi_n) \rightarrow 0$ .

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